Extending the Real Business Cycle Model Part 1

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Macroeconomics II

Goals

1. The RBC model provides compelling explanations for business cycles but does not match some aspects of the data.
   - We are going to extend the model in hope to match the data better.
2. So far, the model still abstracts from several characteristics that characterize a modern economy.
   - We will introduce money into the economy.
   - We will introduce imperfect competition into the economy.

The cyclical behavior of hours

The cyclical behavior of hours

1. We have seen that hours are not volatile enough.
2. Moreover, hours are too strongly correlated with wages.
3. We start by considering modifications that will improve the fit of the model
   - Change the utility specification.
   - Adding labor supply shocks.
4. We will also address a further issue of hours worked: their impulse response to a technology shock.

Intratemporal non-separability

To overcome the strong wealth effect, Greenwood et al. (1988) suggest the following preference specification:

$$U(C_t, H_t) = \frac{\left(C_t - \phi \frac{H_t^{1+\eta}}{1+\eta}\right)^{1-\gamma}}{1-\gamma}. \tag{1}$$

The utility from consumption and hours are not additive separable. We will see that this particular specification eliminates the wealth effect.

The household problem

The household solves:

$$\max_{C_t, K_{t+1}, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{\left(C_t - \phi \frac{H_t^{1+\eta}}{1+\eta} \right)^{1-\gamma}}{1-\gamma} \right\} \tag{2}$$

s.t.

$$C_t + K_{t+1} = W_t H_t + R_t K_t + \Pi_t + (1-\delta) K_t$$
$$I_t = K_{t+1} - (1-\delta) K_t$$

First order conditions

The Lagrangian is:

$$\Lambda_t = \mathbb{E}_0 \Big\{ \sum_{t=0}^{\infty} \beta^t \Big[\frac{\left(C_t - \phi \frac{H_t^{1+\eta}}{1+\eta}\right)^{1-\gamma}}{1-\gamma} - \lambda_t [C_t + K_{t+1} - W_t H_t - R_t K_t - \Pi_t - (1-\delta) K_t] \Big] \Big\}. \quad (3)$$

$$\frac{\partial \Lambda_t}{\partial C_t} : \left(C_t - \phi \frac{H_t^{1+\eta}}{1+\eta}\right)^{-\gamma} = \lambda_t \quad (4)$$

$$\frac{\partial \Lambda_t}{\partial K_{t+1}} : \beta^t \lambda_t = \mathbb{E}_t \Big\{ \beta^{t+1} \lambda_{t+1} \left(R_{t+1} + (1-\delta)\right) \Big\} \quad (5)$$

$$\frac{\partial \Lambda_t}{\partial H_t} : \phi H_t^{\eta} \left(C_t - \phi \frac{H_t^{1+\eta}}{1+\eta}\right)^{-\gamma} = \lambda_t W_t \quad (6)$$

Optimality

$$\left(C_t - \phi \frac{H_t^{1+\eta}}{1+\eta}\right)^{-\gamma} = \lambda_t$$

The marginal utility of consumption still needs to equal the marginal cost of consumption. However, the marginal utility of consumption now depends on the number of hours worked in the same period. When hours worked are high, the marginal utility of consumption is high. Hence, when the household wants to consume a lot (a boom) it also wants to work a lot.

The Euler equation:

$$\left(C_t - \phi \frac{H_t^{1+\eta}}{1+\eta}\right)^{-\gamma} = \mathbb{E}_t \left\{ \beta \left(C_{t+1} - \phi \frac{H_{t+1}^{1+\eta}}{1+\eta}\right)^{-\gamma} (R_{t+1} + (1-\delta)) \right\}.$$

Optimality II

Optimal hours:

$$H_t = \left(\frac{1}{\phi} W_t\right)^{\frac{1}{\eta}}. \tag{7}$$

The marginal disutility of work also depends on the marginal utility of consumption. This disutility needs to equal the marginal benefit which is the wage times the marginal utility of consumption. As a result, the optimal hours choice depend only and positively on wages. There is no off-setting wealth effect.

Balanced growth

$$H_t = \left(\frac{1}{\phi} W_t\right)^{\frac{1}{\eta}} \tag{8}$$

- Note that this preference specification is inconsistent with balanced growth.
- W_t has sky-rocked over the last 100 years; hours have, if any, decreased.
- Hence, we need to assume that households respond to temporary technological disturbances different than to permanent technological change.

Solving for the steady state

$$W^{ss} = (1-\alpha)(k^{ss})^{\alpha} \tag{9}$$

$$R^{ss} = \frac{1}{\beta} - 1 + \delta \tag{10}$$

$$\frac{C^{ss}}{H^{ss}} = (k^{ss})^{\alpha} - \delta k^{ss} \tag{11}$$

$$\frac{I^{ss}}{H^{ss}} = \delta k^{ss} \tag{12}$$

$$\frac{Y^{ss}}{H^{ss}} = (k^{ss})^{\alpha} \tag{13}$$

$$\frac{K^{ss}}{H^{ss}} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}\right)^{\frac{1}{1-\alpha}} \tag{14}$$

$$H^{ss} = \left[\frac{1}{\phi} W^{ss}\right]^{\frac{1}{\eta}}. \tag{15}$$

Results

	Y	C	I	H	TFP	w	r
				Data			
Std. %	1.61	1.25	7.27	1.9	1.25	0.96	1.02
				$\eta = 0.5$			
Std. %	1.56	0.45	5.30	0.52	1.24	1.10	0.06
				$\eta = 0.001$			
Std. %	1.99	0.48	7.09	1.19	1.24	0.96	0.07
				GHH $\eta = 0.5$			
Std. %	2.23	1.57	4.37	1.48	1.24	0.74	0.08

Discussion

Improvements:

- Even with a defendable labor supply elasticity, hours are now twice as volatile as wages.
- Output and consumption become more volatile.

Deterioration:

- Wages are not volatile enough. As hours are strongly procyclical, MPL becomes less volatile.

Labor supply shocks

- The baseline model has only one shock (labor productivity) that drives all macroeconomic aggregates.
- A shock that preserves the co-movement between macroeconomic quantities but breaks the co-movement between quantities and prices would be nice.
- Shocks to the preferences in leisure appear especially promising because they break the tight link between hours and wages.
- Unfortunately, it is hard to see what these shocks are supposed to represent.

Preference shocks

- We model these shocks as changing the disutility to work:

$$\phi \Omega_t \frac{H_t^{1+\eta}}{1+\eta} \tag{16}$$

$$\ln \Omega_{t+1} = \rho_\omega \ln \Omega_t + \omega_{t+1} \tag{17}$$

- Put differently, sometimes work is more painful than other times.

- We will assume a normal distribution for preference shocks and that these shocks are independent of other technology shocks:

$$\omega_{t+1} \sim N(0, \sigma_\omega^2) \tag{18}$$

$$Cov(\omega_{t+1}, \epsilon_{t+1}) = 0. \tag{19}$$

The household problem

The household problem becomes:

$$\max_{C_t, K_{t+1}, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \Omega_t \frac{H_t^{1+\eta}}{1+\eta} \right) \right\} \tag{20}$$

s.t.

$$C_t + K_{t+1} = W_t H_t + R_t K_t + \Pi_t + (1-\delta) K_t \tag{21}$$

$$I_t = K_{t+1} - (1-\delta) K_t \tag{22}$$

$$\ln \Omega_{t+1} = \rho_\omega \ln \Omega_t + \omega_{t+1} \tag{23}$$

First order conditions

The Lagrangian is:

$$\Lambda_t = \mathbb{E}_0 \Big\{ \sum_{t=0}^{\infty} \beta^t \Big[\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \Omega_t \frac{H_t^{1+\eta}}{1+\eta} \right) - \lambda_t [C_t + K_{t+1} - W_t H_t - R_t K_t - \Pi_t - (1-\delta) K_t] \Big] \Big\}. \quad (24)$$

$$\frac{\partial \Lambda_t}{\partial C_t} : C_t^{-\gamma} = \lambda_t \quad (25)$$

$$\frac{\partial \Lambda_t}{\partial K_{t+1}} : \beta^t \lambda_t = \mathbb{E}_t \Big\{ \beta^{t+1} \lambda_{t+1} \left(R_{t+1} + (1-\delta) \right) \Big\} \quad (26)$$

$$\frac{\partial \Lambda_t}{\partial H_t} : \phi \Omega_t H_t^{\eta} = \lambda_t W_t \quad (27)$$

Intratemporal optimality

The first order conditions are unchanged but the optimal labor supply:

$$\phi \Omega_t H_t^{\eta} = C_t^{-\gamma} W_t \tag{28}$$

The marginal disutility of working = the marginal gain from working (marginal utility of consumption times the wage rate).

However, now the marginal disutility of working depends on Ω_t.

The cyclical movement of hours and wages

$$\phi \Omega_t H_t^{\eta} = C_t^{-\gamma} W_t \tag{29}$$

Log-linearize using **LI Rule 1** and **LI Rule 4** yields

$$(H^{ss})^{\eta}(1 + \hat{\Omega}_t + \eta \hat{H}_t) = (C^{ss})^{-\gamma} W^{ss} \frac{1}{\phi}(1 - \gamma \hat{C}_t + \hat{W}) \tag{30}$$

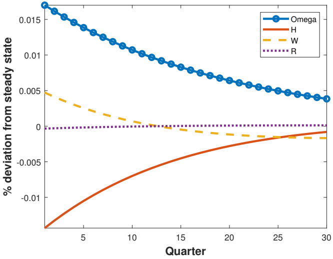
$$\hat{H}_t = \frac{1}{\eta}[-\gamma \hat{C}_t - \hat{\Omega}_t + \hat{W}_t]. \tag{31}$$

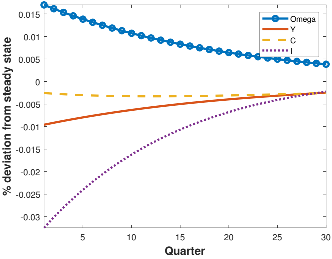
- Fluctuations in hours depend now on preference shocks.
- These shocks have the exact same effect as changes in wages.

Calibration

- We are going to use the same calibration as before with an inverse labor supply elasticity $\eta = 0.5$.
- I choose the autocorrelation of leisure shocks to match the autocorrelation of hours in the data. This leads to $\rho_\omega = 0.95$.
- I use their standard deviation to match the standard deviation of hours in the data. This leads to $\sigma_\omega = 0.017$.

Impulse response functions





- After an increase in the utility of leisure, hours worked decline.
- As a result, wages increase and the interest rate and output decline.
- The household responds by a small fall in consumption and a large decline in investment.

Comparing models and data

	Y	C	I	H	TFP	w	r
				Data			
Std. %	1.61	1.25	7.27	1.9	1.25	0.96	1.02
				$\sigma_\omega = 0$			
Std. %	1.56	0.45	5.30	0.52	1.24	1.10	0.06
				$\sigma_\omega = 0.017$			
Std. %	2.00	0.57	6.79	1.94	1.24	1.27	0.07

Correlations

	Y	C	I	H	TFP	w	r
				Data			
Y	1						
C	0.78	1					
I	0.83	0.67	1				
H	0.87	0.68	0.76	1			
TFP	0.79	0.71	0.77	0.49	1		
w	0.12	0.29	0.07	-0.06	0.34	1	
r	0.24	0.12	0.20	0.40	0.05	-0.13	1
				$\sigma_\omega = 0.017$			
Y	1						
C	0.94	1					
I	1	0.91	1				
H	0.79	0.70	0.80	1			
TFP	0.78	0.73	0.78	0.25	1		
w	0.37	0.42	0.35	-0.28	0.85	1	
r	0.96	0.81	0.98	0.80	0.75	0.29	1

Discussion

Improvements:

- We match now the volatility of hours.
- The economy in general is more volatile.
- The correlation between TFP and other aggregates is no longer one.
- Wages are close to acyclical and somewhat negatively correlated with hours.
- Also the size of the correlations between other aggregates is now better.

Deterioration:

- The correlation between TFP and hours is somewhat too low.

Discussion II

- In general, the fit is very much improved.
- The main issue is that we can only identify preference shocks indirectly and have no idea what they are.
- But the exercise tells us what an alternative, more tangible mechanism should have properties that are somewhat similar to preference shocks to explain the data.

"Alternatively, one could explain the observed pattern without a procyclical real wage by positing that tastes for consumption relative to leisure vary over time. Recessions are then periods of "chronic laziness." As far as I know, no one has seriously proposed this explanation of the business cycle." Mankiw (1989)

Adjustment costs

Using ▸ impulse response functions, Gali (1999) shows that hours initially decline after a positive productivity shock which does not hold in our model this far. One way to fix this issues is to introduce investment adjustment costs. The idea is that large changes in the capital stock disrupt the production process and, hence, are costly:

$$K_{t+1} = I_t - \frac{\psi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t.$$

- This is our former model with $\psi = 0$.

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$$K_{t+1} = I_t - \frac{\psi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t.$$

- This is our former model with $\psi = 0$.
- Investment adjustment costs are zero when only depreciated capital is replaced.

The household problem

The household problem now becomes:

$$\max_{C_t, I_t, H_t, K_{t+1}} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) \right\} \tag{32}$$

s.t.

$$C_t + I_t = W_t H_t + R_t K_t + \Pi_t \tag{33}$$

$$I_t = K_{t+1} - (1-\delta)K_t + \frac{\psi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t. \tag{34}$$

Instead of substituting for investment, I will write the problem with two constraints.

First order conditions I

The Lagrangian is:

$$\Lambda_t = \mathbb{E}_0\Big\{\sum_{t=0}^{\infty}\beta^t\Big[\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi\frac{H_t^{1+\eta}}{1+\eta}\right) - \lambda_t[C_t - W_t H_t - R_t K_t - \Pi_t + I_t] + \mu_t[I_t + (1-\delta)K_t - \frac{\psi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t - K_{t+1}]\Big]\Big\}. \quad (35)$$

$$\frac{\partial\Lambda_t}{\partial C_t} : C_t^{-\gamma} = \lambda_t \quad (36)$$

$$\frac{\partial\Lambda_t}{\partial H_t} : \phi H_t^{\eta} = \lambda_t W_t \quad (37)$$

First order conditions II

The first order conditions for investment and the capital choice:

$$\Lambda_t = \mathbb{E}_0\Big\{ \sum_{t=0}^{\infty} \beta^t \Big[\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) - \lambda_t [C_t - W_t H_t - R_t K_t - \Pi_t + I_t] + \mu_t [I_t + (1-\delta) K_t - \frac{\psi}{2} \left(\frac{I_t}{K_t} - \delta \right)^2 K_t - K_{t+1}] \Big] \Big\}. \tag{38}$$

$$\frac{\partial \Lambda_t}{\partial I_t} : \lambda_t = \mu_t \left[1 - \psi \left(\frac{I_t}{K_t} - \delta \right) \right] \tag{39}$$

$$\frac{\partial \Lambda_t}{\partial K_{t+1}} : \beta^t \mu_t = \beta^{t+1} \mathbb{E}_t \Big\{ \lambda_{t+1} R_{t+1} + \mu_{t+1} \Big[(1-\delta) - \frac{\psi}{2} \left(\frac{I_{t+1}}{K_{t+1}} - \delta \right)^2 + \psi \frac{I_{t+1}}{K_{t+1}} \left(\frac{I_{t+1}}{K_{t+1}} - \delta \right) \Big] \Big\}. \tag{40}$$

The value of capital

Note, λ_t is the marginal value of one more unit of consumption. Similarly, μ_t is the marginal value of having one more unit of installed capital. Hence, using (39),

$$q_t = \frac{\mu_t}{\lambda_t} = \left[1 - \psi\left(\frac{I_t}{K_t} - \delta\right)\right]^{-1} \tag{41}$$

is the amount of consumption the household is willing to give up for one more unit of installed capital.

- With $\psi = 0$, $q_t = 1$, i.e., the household is always exactly indifferent between consumption and one more unit of installed capital.
- This is not true with adjustment costs.
- After a positive technology shock, $I_t > \delta K_t$ and, hence, $q_t > 1$.
- The value of installed capital is procyclical.

Rewrite FOCs in q_t

Rewriting the first order condition for capital yields:

$$q_t = \mathbb{E}_t\Big\{\beta\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\Big(R_{t+1} + q_{t+1}\Big[(1-\delta) - \frac{\psi}{2}\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)^2 + \psi\frac{I_{t+1}}{K_{t+1}}\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)\Big)\Big]\Big\}. \quad (42)$$

The current value of installed capital in terms of the consumption good needs to equal the discounted expected return on capital tomorrow plus the expected value of installed capital tomorrow in terms of the consumption good times a function of depreciation and adjustment costs.

The firm problem

The firm problem is the same as before:

$$\max_{K_t, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{-\gamma}}{C_0^{-\gamma}} \left[K_t^{\alpha} \left(A_t H_t \right)^{1-\alpha} - W_t H_t - R_t K_t \right] \right\} \tag{43}$$

With FOCs:

$$R_t = \alpha K_t^{\alpha-1} \left(A_t H_t \right)^{1-\alpha} \tag{44}$$

$$W_t = (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} H_t^{-\alpha}. \tag{45}$$

Solution to the model I

We can now summarize the equations characterizing the equilibrium. We start with the budget constraint and variable definitions:

$$C_t + I_t = Y_t \tag{46}$$

$$I_t = K_{t+1} - (1-\delta)K_t + \frac{\psi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t \tag{47}$$

$$Y_t = K_t^{\alpha}\left(A_t H_t\right)^{1-\alpha} \tag{48}$$

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1}. \tag{49}$$

Solution to the model II

Optimality conditions:

$$\phi H_t^{\eta} = C_t^{-\gamma} W_t \tag{50}$$

$$q_t = \left[1 - \psi\left(\frac{I_t}{K_t} - \delta\right)\right]^{-1} \tag{51}$$

$$q_t = \mathbb{E}_t \beta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \left\{ R_{t+1} + q_{t+1}\left[(1-\delta) - \frac{\psi}{2}\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)^2 + \psi \frac{I_{t+1}}{K_{t+1}}\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)\right]\right\} \tag{52}$$

$$R_t = \alpha K_t^{\alpha-1}\left(A_t H_t\right)^{1-\alpha} \tag{53}$$

$$W_t = (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} H_t^{-\alpha}. \tag{54}$$

Which gives us 9 equations in 9 unknowns (C_t, H_t, K_t, I_t, Y_t, A_t, q_t, W_t, R_t).

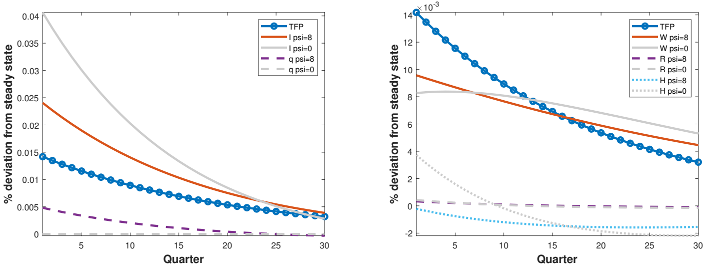
Solution and calibration

- Note, we are treating q_t as any other endogenous variable. Alternatively, we could have substituted for investment in (33), or we could have written the problem in terms of the Lagrange multiplier μ_t. Writing the problem in terms of q_t has the advantage that it is more easily interpretable.

- We are going to use the same calibration as before with an inverse labor supply elasticity $\eta = 0.5$.

- I set $\psi = 8$ which is arbitrary.

Impulse response functions



- Because of adjustment costs, investment responds less to an increase in productivity.
- As consumption rises by more, **hours decrease on impact**.
- Consequently, the interest rate increases by less.

Comparing models and data

	Y	C	I	H	TFP	w	r
				Data			
Std. %	1.61	1.25	7.27	1.9	1.25	0.96	1.02
				$\psi = 0$			
Std. %	1.56	0.45	5.30	0.52	1.24	1.10	0.06
				$\psi = 8$			
Std. %	1.22	0.63	3.14	0.08	1.24	1.25	0.04

Correlations

	Y	C	I	H	TFP	w	r
				Data			
Y	1						
C	0.78	1					
I	0.83	0.67	1				
H	0.87	0.68	0.76	1			
TFP	0.79	0.71	0.77	0.49	1		
w	0.12	0.29	0.07	-0.06	0.34	1	
r	0.24	0.12	0.20	0.40	0.05	-0.13	1
				$\psi = 8$			
Y	1						
C	1	1					
I	1	0.99	1				
H	-0.35	-0.43	-0.3	1			
TFP	1	0.99	1	-0.32	1		
w	1	1	1	-0.41	1	1	
r	0.97	0.95	0.99	-0.3	0.98	0.96	1

Discussion

Improvements:

- Hours respond initially negatively to productivity shocks.
- Hours and wages are negatively correlated.
- Consumption is more volatile.

Deterioration:

- Hours are now countercyclical.
- The economy shows yet less volatility than without adjustment costs.
- Particularly investment and hours are not volatile enough.

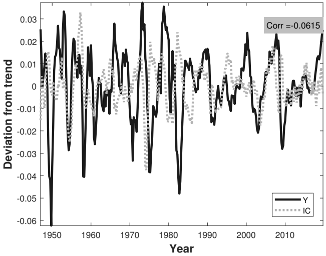
General methods of moments (GMM)

Too low investment volatility

The idea

- The baseline RBC model features too little volatility in the investment rate.
- Greenwood et al. (1997) find that relative changes in the price of investment is a major source of long-run economic growth.
- For example, over the last 30 years, the price of computing power has declined considerably while prices of most other goods have increased.
- When investment becomes relatively cheaper, it is optimal to invest more in capital leading to higher long run output.
- There is again no reason to believe that the process of investment prices is deterministic.

The idea II



- Fisher (2006) asks whether shocks to the price of investment explain business cycle fluctuations.

- The figure displays the cost of investment relative to consumption. Indeed, the relative price of investment shows large fluctuations at the business cycle frequency.

Investment-specific shocks

- We model these shocks as efficiency in converting investment into capital:

$$K_{t+1} = Z_t I_t + (1-\delta)K_t \tag{55}$$

$$\ln Z_{t+1} = \rho_z \ln Z_t + z_{t+1} \tag{56}$$

- We will assume a normal distribution for investment-specific shocks and that these shocks are independent of other technology shocks:

$$z_{t+1} \sim N(0, \sigma_z^2) \tag{57}$$

$$Cov(z_{t+1}, \epsilon_{t+1}) = 0. \tag{58}$$

- Note, you may also think about these shocks as financial sector shocks where the efficiency of the sector to allocate capital is fluctuating.

The household problem

The household solves:

$$\max_{C_t, K_{t+1}, H_t} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi \frac{H_t^{1+\eta}}{1+\eta} \right) \right\} \tag{59}$$

s.t.

$$C_t + I_t = W_t H_t + R_t K_t + \Pi_t \tag{60}$$

$$K_{t+1} = Z_t I_t + (1-\delta) K_t \tag{61}$$

$$\ln Z_{t+1} = \rho_z \ln Z_t + z_{t+1}. \tag{62}$$

First order conditions

The Lagrangian is:

$$\Lambda_t = \mathbb{E}_0\Big\{\sum_{t=0}^{\infty}\beta^t\Big[\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \phi\frac{H_t^{1+\eta}}{1+\eta}\right) - \lambda_t[C_t + \frac{K_{t+1}}{Z_t} - W_tH_t - R_tK_t - \Pi_t - (1-\delta)\frac{K_t}{Z_t}]\Big]\Big\}. \tag{63}$$

$$\frac{\partial\Lambda_t}{\partial C_t} : C_t^{-\gamma} = \lambda_t \tag{64}$$

$$\frac{\partial\Lambda_t}{\partial K_{t+1}} : \beta^t\frac{\lambda_t}{Z_t} = \mathbb{E}_t\Big\{\beta^{t+1}\lambda_{t+1}\left(R_{t+1} + \frac{(1-\delta)}{Z_{t+1}}\right)\Big\} \tag{65}$$

$$\frac{\partial\Lambda_t}{\partial H_t} : \phi H_t^{\eta} = \lambda_t W_t. \tag{66}$$

Intertemporal optimality

The optimal hours decision is unchanged. The Euler equation becomes:

$$C_t^{-\gamma} = \mathbb{E}_t \left\{ \beta C_{t+1}^{-\gamma} \frac{Z_t}{Z_{t+1}} \left(Z_{t+1} R_{t+1} + (1-\delta) \right) \right\} \tag{67}$$

The intertemporal decision now takes into account that one unit of investment today costs $\frac{1}{Z_t}$, and tomorrow it costs $\frac{1}{\mathbb{E}_t Z_{t+1}}$.

Log-linearization

The log-linearized Euler equation is:

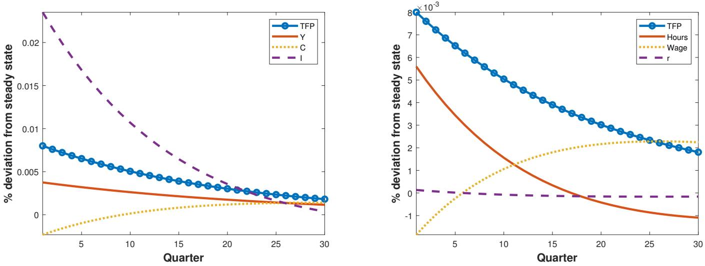
$$\mathbb{E}_t \hat{C}_{t+1} - \hat{C}_t = \frac{1}{\gamma}\left[\hat{Z}_t - \hat{Z}_{t+1} + (1-\beta(1-\delta))[\mathbb{E}_t \hat{Z}_{t+1} + \mathbb{E}_t \hat{R}_{t+1}]\right]. \quad (68)$$

Consumption growth is high when investing today is cheap relative to the expected cost of investment tomorrow, i.e., households are willing to defer consumption when investment is cheap.

Calibration

- We are going to use the same calibration as before with an inverse labor supply elasticity $\eta = 0.5$.
- I choose the autocorrelation of investment shocks to match the autocorrelation of the data. This leads to $\rho_z = 0.95$.
- I use their standard deviation to match the standard deviation in the data. This leads to $\sigma_z = 0.80\%$.

Impulse response functions



- An increase in the efficiency of investment leads to a strong investment boom.
- Consumption declines on impact because households defer consumption.
- Hence, hours increase because of the wealth effect. Hence, wages decrease.

Comparing models and data

	Y	C	I	H	TFP	w	r
				Data			
Std. %	1.61	1.25	7.27	1.9	1.25	0.96	1.02
				$\sigma_z = 0$			
Std. %	1.56	0.45	5.30	0.52	1.24	1.10	0.06
				$\sigma_z = 0.80\%$			
Std. %	1.64	0.55	6.13	0.91	1.24	1.14	0.06

Correlations

	Y	C	I	H	TFP	w	r
				Data			
Y	1						
C	0.78	1					
I	0.83	0.67	1				
H	0.87	0.68	0.76	1			
TFP	0.79	0.71	0.77	0.49	1		
w	0.12	0.29	0.07	-0.06	0.34	1	
r	0.24	0.12	0.20	0.40	0.05	-0.13	1
				$\sigma_z = 0.80$			
Y	1						
C	0.57	1					
I	0.97	0.35	1				
H	0.74	-0.13	0.88	1			
TFP	0.95	0.76	0.86	0.53	1		
w	0.85	0.92	0.69	0.27	0.95	1	
r	0.94	0.42	0.95	0.79	0.90	0.72	1

Discussion

Improvements:

- The economy is more volatile, especially investment and hours worked.
- Wages and hours are only weakly correlated.
- *TFP* is no longer perfectly correlated with other aggregates.

Deterioration:

- Consumption is not procyclical enough.

Appendix

Impulse response functions in the data

Impulse response functions in the data

- We have already seen how to compute impulse response functions in the model.
- Key for this was that we know the structural shock.
- In the data, usually, we cannot observe the shock. Instead, we need to estimate it first.

A VAR representation

Suppose there exists a linear relationship between those variables and their current and past realizations and economic shocks. Recall that this is true in our log-linearized model:

$$A_1 \begin{bmatrix} \mathbb{E}_t X_{t+1} \\ \mathbb{E}_t Y_{t+1} \end{bmatrix} = A_0 \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + aZ_{t+1}, \tag{69}$$

To simplify the notation, consider the case with two variables, $\mathbf{y}_t = [y_{1,t} \quad y_{2,t}]'$:

$$B \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \Gamma_0 + \Gamma_1 \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}, \tag{70}$$

$$B = \begin{bmatrix} 1 & b_{1,2} \\ b_{2,1} & 1 \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} b_{1,0} \\ b_{2,0} \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{bmatrix}. \tag{71}$$

The reduced form VAR

In the data, we can estimate

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = B^{-1}\Gamma_0 + B^{-1}\Gamma_1 \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + B^{-1} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \tag{72}$$

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = C_0 + C_1 \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}. \tag{73}$$

Note, we have

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & b_{1,2} \\ b_{2,1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}. \tag{74}$$

Hence, the reduced-form errors are different from the structural errors. In particular, they are a linear combination of those structural errors.

The issue of estimation

- Can we estimate the reduced form model and recuperate the structural parameters? This would allow us to identify shocks and compute impulse response functions to these shocks.
- In general, the answer to this question is no.
- The issue is an identification problem. Because of the contemporaneous effect of the variables on each other, the structural model has more parameters than the reduced-form model.
- In our example, the structural model has 10 parameters, and the reduced form has only 9.
- We need to impose a restriction on the model to identify it. Economics, over the time has come up with a series of restrictions grounded in economic theory. A typical restriction is one on timing.

Cholesky decomposition

- The idea is to assume that one of our endogenous variables is predetermined and not affected by contemporaneous changes in the other variable.
- In our model, for example, capital is predetermined and not affected by output today.
- The reasonableness of this assumption depends on the frequency of your data.

With $b_{2,1} = 0$ **we have**

$$B^{-1} = \begin{bmatrix} 1 & b_{1,2} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -b_{1,2} \\ 0 & 1 \end{bmatrix}. \quad (75)$$

Identification

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & -b_{1,2} \\ 0 & 1 \end{bmatrix} \Gamma_0 + \begin{bmatrix} 1 & -b_{1,2} \\ 0 & 1 \end{bmatrix} \Gamma_1 \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} 1 & -b_{1,2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \tag{76}$$

$$\begin{aligned} \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = & \begin{bmatrix} b_{1,0} - b_{1,2} b_{2,0} \\ b_{2,0} \end{bmatrix} \\ & + \begin{bmatrix} \gamma_{1,1} - b_{1,2}\gamma_{2,1} & \gamma_{1,2} - b_{1,2}\gamma_{2,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} 1 & -b_{1,2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \end{aligned} \tag{77}$$

Identification II

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} b_{1,0} - b_{1,2}b_{2,0} \\ b_{2,0} \end{bmatrix} + \begin{bmatrix} \gamma_{1,1} - b_{1,2}\gamma_{2,1} & \gamma_{1,2} - b_{1,2}\gamma_{2,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} 1 & -b_{1,2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \tag{78}$$

$$\epsilon_{2,t} = u_{2,t} \qquad \sigma_2^2 = Var(u_{2,t})$$

$$b_{1,2} = -\frac{COV(u_{1,t}, u_{2,t})}{\sigma_2^2} \qquad \epsilon_{1,t} = b_{1,2}\epsilon_{2,t} + u_{1,t}$$

$$\sigma_1^2 = Var(u_{1,t}) + b_{1,2}^2\sigma_2^2 \qquad b_{2,0} = c_{2,0}$$

$$\gamma_{2,2} = c_{2,2} \qquad \gamma_{2,1} = c_{2,1}$$

$$\gamma_{1,2} = c_{1,2} + b_{1,2}\gamma_{2,2} \qquad \gamma_{1,1} = c_{2,1} + b_{1,2}\gamma_{2,1}$$

$$b_{1,0} = c_{1,0} + b_{1,2}b_{2,0}$$

Estimation in practice

- *Matlab* allows you to compute VAR models and impulse response functions.
- However, I find *Eviews* more user friendly for that purpose.
- Note, we require all variables to be stationary.
- Our structural models usually imply that an $VAR(1)$ model is sufficient. However, in the data, we may want to be less restrictive.
- How to select the order of the VAR model? One popular approach is to minimize the Akaike information criteria:

$$AIC = 2k - 2\ln(\hat{L}). \tag{79}$$

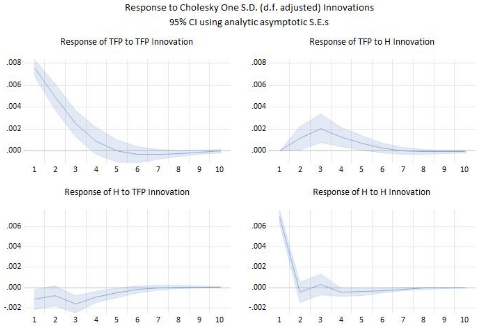
Application: productivity and hours

- Consider a simple bi-variate VAR of labor productivity (LP) and hours.
- The variables are non-stationary. To make them stationary, we use first-differences of the logs.

$$\begin{bmatrix} \Delta \ln H_t \\ \Delta \ln LP_t \end{bmatrix} = C_0 + C_1 \begin{bmatrix} \Delta \ln H_{t-1} \\ \Delta \ln LP_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}. \tag{80}$$

To identify the model, we will assume that the structural shock on hours, i.e., a labor supply shock, has no contemporaneous effect on TFP, $b_{2,1} = 0$.

Application: productivity and hours II



- Shocks to labor productivity have a persistent effect on labor productivity. Similarly, shocks to hours have a persistent effect on hours.
- A positive shock to labor productivity decreases hours worked initially.

Back

GMM estimation

The idea

- I have set the parameter of the adjustment cost function arbitrarily.
- We could calibrate to match a particular moment such as the response of hours in one quarter after a shock.
- Alternatively, we can use more moments and minimize the distance between those moments in the data and the model.
- We are going to use so called General methods of moments (GMM).

General Methods of Moments (*GMM*)

- Suppose you want to estimate a parameter vector p. In our case, this is the investment adjustment costs.
- Suppose our model creates a total of $\mathcal{M}(p)$ moments.
- We choose a subset $M(p)$ for estimation. In our case, the impulse response function of hours to a TFP shock.
- Let $\tilde{p}$ be the true parameters, and $\hat{M}$ be the sample analogous to M. If our model is correct:

$$\mathbb{E}(\hat{M}(\tilde{p}) - M(\tilde{p})) = 0.$$

GMM Estimation

- For a particular parameter vector p, we can compute the moments generated by our model. In our case, we need to solve the model and compute the impulse response function.

- GMM performs:

$$p = \underset{p}{argmin}((M(p) - \hat{M}(p))W(M(p) - \hat{M}(p))')$$

  W is an appropriate (positive-definite) weighting matrix.

- When number of moments equal to number of parameters, we have exact identification: MM.

  Having more moments increases efficiency.

  Allows us to test our overidentified model.

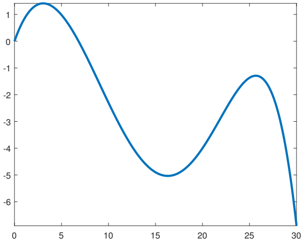
Weighting Matrix

- Often, studies use the identity weighting matrix.
- We are going to do the same here.
- The identity weighting matrix is, however, not efficient. For example, one can show that it is more efficient to give higher weight to moments estimated with higher precision.

Minimizing the function

- We still need to a routine that solves the minimization problem for us.
- The simplest form is called grid-search. We simply solve the problem for different values of p and choose the minimum.
- Grid-search is relatively inefficient as it requires to evaluate the problem for many guesses of p.
- When the loss function is well-behaved, more efficient methods exist. We will use the Matlab function *fmincon*.

Global vs. Local Solutions



- Minimizers are usually designed to find a local minimum.

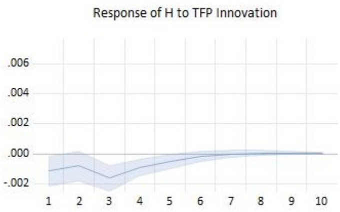
- So called genetic algorithms aim at finding the global minimum:

  Find a local minimum, try other starting values and recompute local minimum.

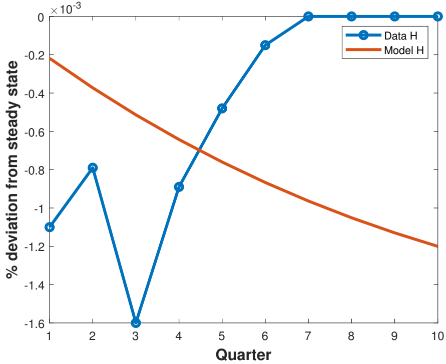
  Pattern search, *simulated annealing*.

Application to investment-adjustment costs

- In general, we can estimate more than one parameter. For simplicity, I fix all other parameters and only estimate the adjustment costs.
- We have to decide on the moments we want to target. I will use the impulse response function for 10 quarters of hours to a shock in TFP.



Result



×10-3
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
% deviation from steady state
1
2
3
4
5
6
7
8
9
10
Quarter
Data H
Model H

▸ Back

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